Take $\mathfrak{g} = sl_2(\mathbb{C})$ and recall the KZ-eq:

$$\kappa \frac{\partial}{\partial z_i} \Psi = \left(\sum_{\substack{j=1 \\ j\neq i}}^{N} \frac{\Omega_{ij}}{z_i - z_j} \right) \Psi \, , \quad i=1, \dots, N$$

Take $\quad V = V_{\mu_1} \otimes V_{\mu_2} \otimes \cdots \otimes V_{\mu_N}$,

where V_{μ_i} is the lowest-weight Verma module over $\mathfrak{g}$ with lowest weight $-\mu_i$.

We have

$$\kappa = K + h^\vee$$

Here we consider solutions with values in finite-dimensional spaces

$$W = (V^{n^-})^\lambda, \quad \lambda = -\sum_{i=1}^{N} \mu_i + \mu, \; \mu \in \bigoplus \mathbb{Z}_+ \alpha_i$$

λ is eigenvalue under diagonal action with H: $\quad Hw = \lambda w, \quad w \in W$

To simplify the discussion and to avoid having to deal with null-vectors, we take $\kappa \notin \mathbb{Q}$.

Recall: $\quad \Omega = E \otimes F + F \otimes E + \frac{1}{2} H \otimes H$

Let us consider $N=3$ (solutions in the tensor product of 3 spaces)

Proposition 5:

Any solution $\Psi(z_1, z_2, z_3)$ of KZ eq. can be written in the form

$$\Psi(z_1, z_2, z_3) = (z_1 - z_2)^{(\Omega_{12}+\Omega_{13}+\Omega_{23})/\kappa} f\left(\frac{z_1 - z_2}{z_1 - z_3}\right),$$

where $f(z)$ satisfies:

$$\kappa \frac{\partial}{\partial z} f(z) = \left(\frac{\Omega_{12}}{z} + \frac{\Omega_{23}}{z-1}\right) f(z) \quad (*)$$

Proof:

Introduce the variables

$$x = \frac{z_1 - z_2}{z_1 - z_3}, \quad y = z_1 - z_3, \quad t = z_1 + z_2 + z_3$$

$\rightarrow$ KZ equations take the form

$$\kappa \frac{\partial \Psi}{\partial x} = \left(\frac{\Omega_{12}}{x} + \frac{\Omega_{23}}{x-1}\right)\Psi,$$

$$\kappa \frac{\partial \Psi}{\partial y} = \left(\frac{\Omega_{12} + \Omega_{13} + \Omega_{23}}{y}\right)\Psi,$$

$$\kappa \frac{\partial \Psi}{\partial t} = 0.$$

Since $\Omega_{12}+\Omega_{13}+\Omega_{23}$ commutes with Ω_{ij}, this shows that the function

$$f = y^{-(\Omega_{12}+\Omega_{13}+\Omega_{23})/k}\, \Psi(x,y)$$

depends only on x and satisfies eq. $(*)$.

$\square$

Let us now consider the case $\mu = 2$, then $\dim W = 2$. If we assume that $\mu_i \neq 0$, then a basis of W is given by:

$$\omega_1 = \mu_2 E v_1 \otimes v_2 \otimes v_3 - \mu_1 v_1 \otimes E v_2 \otimes v_3,$$

$$\omega_2 = \mu_3 v_1 \otimes E v_2 \otimes v_3 - \mu_2 v_1 \otimes v_2 \otimes E v_3.$$

<u>Proposition 6:</u>

Any solution $f(z)$ of $(*)$ can be written as follows:

$$f(z) = z^{\frac{\mu_1\mu_2 - 2\mu_1 - 2\mu_2}{2k}} (1-z)^{\frac{\mu_2\mu_3}{2k}} \left(F(z)\,\omega_1 + z\frac{k}{\mu_3} F'(z)\,\omega_2\right)$$

where $F(z)$ is a solution of the Gauss hypergeometric equation

$$z(1-z)\frac{d^2F}{dz^2} + [c-(a+b+1)z]\frac{dF}{dz} - abF = 0 \qquad (**)$$

with $a = \mu_3/k$, $b = -\mu_1/k$, $c = 1-(\mu_1+\mu_2)/k$.

Proof:

Explicit calculation shows that the action of Ω_{12}, Ω_{23} in the basis ω_1, ω_2 is given by

$$\Omega_{12}\,\omega_1 = \left(\frac{1}{2}\mu_1\mu_2 - \mu_1 - \mu_2\right)\omega_1,$$

$$\Omega_{12}\,\omega_2 = \mu_3\omega_1 + \frac{1}{2}\mu_1\mu_2\omega_2,$$

$$\Omega_{23}\,\omega_1 = \frac{1}{2}\mu_2\mu_3\omega_1 + \mu_1\omega_2,$$

$$\Omega_{23}\,\omega_2 = \left(\frac{1}{2}\mu_2\mu_3 - \mu_2 - \mu_3\right)\omega_2.$$

Let us define

$$g(z) = z^{-\frac{\mu_1\mu_2 - 2\mu_1 - 2\mu_2}{2\kappa}}(1-z)^{-\frac{\mu_2\mu_3}{2\kappa}} f(z)$$

Then g satisfies the differential equation

$$\kappa\frac{d}{dz}g(z) = \left(\frac{\Omega'_{12}}{z} + \frac{\Omega'_{23}}{z-1}\right)g(z),$$

where

$$\Omega'_{12} = \Omega_{12} - \frac{1}{2}\left(\frac{1}{2}\mu_1\mu_2 - \mu_1 - \mu_2\right)\mathrm{Id}:$$

$$\omega_1 \mapsto 0, \quad \omega_2 \mapsto \mu_3\omega_1 + (\mu_1+\mu_2)\,\omega_2$$

$$\Omega'_{23} = \Omega_{23} - \frac{1}{2}\mu_2\mu_3\,\mathrm{Id} : \omega_1 \mapsto \mu_1\omega_2,$$

$$\omega_2 \mapsto -(\mu_1+\mu_3)\,\omega_2.$$

$\Rightarrow$ writing $g(z) = F_1(z)\omega_1 + F_2(z)\omega_2$ implies that F_1, F_2 satisfy the following system of differential equations:

$$\kappa \frac{d}{dz} F_1 = \frac{\mu_3}{z} F_2,$$

$$\kappa \frac{d}{dz} F_2 = \frac{\mu_1+\mu_2}{z} F_2 + \frac{\mu_1}{z-1} F_1 - \frac{\mu_2+\mu_3}{z-1} F_2,$$

which reduces the second equation to

$$\frac{\kappa^2}{\mu_3}\left(zF_1'' + F_1'\right) = \frac{\kappa}{\mu_3}(\mu_1+\mu_2) F_1' + \frac{\mu_1}{z-1} F_1$$
$$- \frac{\kappa(\mu_2+\mu_3)}{\mu_3(z-1)} z F_1'.$$

Simplifying this, we get the hyperg. eq. $(**)$. $\square$

In particular, we can take the function F to be the "Gauss hypergeometric" func. ${}_2F_1(a,b,c;z)$ which is the only solution of $(**)$ satisfying the b.c. $F(0)=1$.

In the disk $|z|<1$, it can be represented as

$${}_2F_1(a,b,c;z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{n!\,(c)_n} z^n,$$

$$(a)_n = a(a+1)\cdots(a+n-1).$$

§6. Vertex operators and OPE

Consider the space of conformal blocks for three points: $0 \in \mathbb{C}$, p and ∞

$\longrightarrow$ associate level κ highest weights λ_0, λ and λ_∞

$\longrightarrow$ obtain space of conformal blocks $\mathcal{H}(0, p, \infty; \lambda_0, \lambda, \lambda_\infty^*)$ as the space of multi-linear maps

$$\Psi : H_{\lambda_0} \times H_\lambda \times H_{\lambda_\infty}^\infty \longrightarrow \mathbb{C}$$

invariant under diagonal action of meromorphic functions with values in $\mathfrak{g}$ and poles at $0, p, \infty$.

Consider conformal block bundle

$$\mathcal{E} = \bigcup_{p \in \mathbb{C} \setminus \{0\}} \mathcal{H}(0, p, \infty; \lambda_0, \lambda, \lambda_\infty^*)$$

and let Ψ be a section of $\mathcal{E}$. Introduce bi-linear map

$$\phi(v, z) : H_{\lambda_0} \otimes H_{\lambda_\infty}^* \longrightarrow \mathbb{C}$$

given by

$$\phi(v, z)(u \otimes w) = \Psi(z)(u, v, w)$$

for $u \in H_{\lambda_0}$, $v \in H_\lambda$ and $w \in H^*_{\lambda_\infty}$.

Regard $\phi(v, z)$ as linear operator from H_{λ_0} to H_{λ_∞}. Note $H_\lambda = \prod_{d \geq 0} H_\lambda(d)$, with $H_\lambda(0) = V_\lambda$ (recall: $L_0 H_\lambda(d) = (\Delta_\lambda + d) H_\lambda(d)$)

Then we have the following

<u>Proposition 1:</u>

Let Ψ be a section of the above conformal block bundle $\mathcal{E}$. Then the linear operator $\phi(v, z)$, $v \in V_\lambda$, $z \in \mathbb{C} \setminus \{0\}$, defined by $\phi(v, z)(u \otimes w) = \Psi(z)(u, v, w)$ satisfies the commutation relation

$$[X \otimes t^n, \phi(v, z)] = z^n \phi(Xv, z) \qquad (*)$$

for $X \otimes t^n \in \hat{\mathfrak{g}}$.

<u>Proof:</u>

Consider the meromorphic function $f(z) = X \otimes z^n$, $X \in \mathfrak{g}$, $n \in \mathbb{Z}$. The action of $f(z)$ on H_{λ_0}, V_λ and $H^*_{\lambda_\infty}$ are given by

$$f(z) u = (X \otimes t^n) u, \quad u \in H_{\lambda_0},$$

$$f(z)\,v = z^n X v, \quad v \in V_\lambda,$$
$$f(z)\,w = -w(X\otimes t^n), \quad w \in H^*_{\lambda_\infty}$$

$\longrightarrow$ invariance of Ψ under action of f implies:

$$\Psi((X\otimes t^n)u, v, w) + z^n \Psi(u, Xv, w) - \Psi(u, v, w(X\otimes t^n)) = 0$$

$$\Longleftrightarrow z^n \langle w, \phi(Xv, z)u\rangle$$
$$= \underbrace{\langle w(X\otimes t^n), \phi u\rangle}_{= \langle w, (X\otimes t^n)\phi u\rangle} - \langle w, \phi(X\otimes t^n)u\rangle$$
$$= \langle w, [X\otimes t^n, \phi(v,z)]u\rangle$$

$\square$

Relation $(*)$ is called "gauge invariance"

<u>Definition</u>:

Suppose that Ψ is horizontal section of $\mathcal{E}$ with respect to the connection $\nabla = d - \omega$. Such an operator

$$\Psi(z) : H_{\lambda_0} \otimes H_\lambda \otimes H^*_{\lambda_\infty} \longrightarrow \mathbb{C}$$

is called a "chiral vertex operator". $\phi(v,z), v \in V_\lambda$, is called "primary field". The operators $\phi(v,z)$, $v \in \bigoplus_{d\geq 0} H_\lambda(d)$, are called "descendents".